Summer Math Practice

At St. Andrew’s we hope that you see math as an endeavor that pushes you to be curious, collaborative and creative. We want you to have this very same experience when you are working on the summer math assignment, so we’ve provided the following resources for you to get help and collaborate on your mathematical work this summer.

Here are all the ways we encourage you to get help:

1. **The SAS Math Slack Group** ([https://sasmath.slack.com](https://sasmath.slack.com))—Slack is a group communication tool that makes it very easy for you to ask questions about problems and have those questions be seen by your peers and all of the teachers in the math department. This is the fastest way to get help on a problem.

2. **Google docs for individual assignments**—If you go to the Slack group, you’ll see that we’ve created a Google doc for each summer assignment. In the Google doc, you are welcome to share solutions to problems you solved, and you will find other students have done the same.

3. **The SAS Math Department Facebook page** ([https://www.facebook.com/sasmathdept/](https://www.facebook.com/sasmathdept/))—We’ve created a facebook page to share tips and suggestions about the summer math assignments, other enriching activities you might find interesting, and share news about the department. Please like us!

4. **Bi-weekly Google Hangouts**—every two weeks, a math teacher will hold an open Google hangout session. We’ll post the link to facebook, and you are free to join in and ask any questions you have. We will also record these sessions and post them so that you can watch them if you aren’t able to participate.

5. **Emailing your teachers**. All of your teachers are here to help, and we welcome your questions via email. Some of your teachers might be away from email while they are on vacation, so if your question is urgent, we encourage you to share it on Slack.

Your specific assignment is attached to this cover sheet. It consists of five pages of problems. Please write up any five problems from each page, for a total of twenty-five problems. Please use the opportunities mentioned above for help if you need it. The primary purpose is for you to continue practicing the mathematical thinking that you’ve been doing so well all year long. Enjoy and we’ll see you on Slack!
1. When triangle $ABC$ is similar to triangle $PQR$, with $A$, $B$, and $C$ corresponding to $P$, $Q$, and $R$, respectively, it is customary to write $ABC \sim PQR$. Suppose that $AB = 4$, $BC = 5$, $CA = 6$, and $RP = 9$. Find $PQ$ and $QR$.

2. What is the size of the acute angle formed by the $x$-axis and the line $3x + 2y = 12$?

3. To the nearest tenth of a degree, find the sizes of the acute angles in the right triangle whose long leg is 2.5 times as long as its short leg.

4. Draw a large acute-angled triangle $ABC$. Carefully add the altitudes $AE$ and $BF$ to your drawing. Measure the lengths of $AE, BF, BC$, and $AC$. Where have you seen the equation $(AE)(BC) = (BF)(AC)$ before? What can you say about the right triangles $AEC$ and $BFC$? Justify your response.

5. Let $A = (0, 0), B = (15, 0), C = (5, 8), D = (9, 0),$ and $P = (6, 6)$. Draw triangle $ABC$, segments $CD, PA,$ and $PB$, and notice that $P$ is on segment $CD$. There are now three pairs of triangles in the figure whose areas are in a 3:2 ratio. Find them, and justify your choices.

6. The transformation $T(x, y) = (ax + by, cx + dy)$ sends $(13, 0)$ to $(12, 5)$ and it also sends $(0, 1)$ to $(-\frac{5}{13}, \frac{12}{13})$. Find $a, b, c,$ and $d$, then describe the nature of this transformation.

7. Let $A = (0, 5), B = (-2, 1), C = (6, -1),$ and $P = (12, 9)$. Let $A', B', C'$ be the midpoints of segments $PA, PB,$ and $PC$, respectively. After you make a diagram, identify the center and the magnitude of the dilation that transforms triangle $ABC$ onto $A'B'C'$.

8. One triangle has sides that are 5 cm, 7 cm, and 8 cm long; the longest side of a similar triangle is 6 cm long. How long are the other two sides?

9. Dale is driving along a highway that is climbing a steady 9-degree slope. After driving for two miles along this road, how much altitude has Dale gained? (One mile = 5280 feet.)

10. (Continuation) How far must Dale travel in order to gain a mile of altitude?

11. Show that $P = (3.2, 6.3)$ is not on the line $4x - y = 7$. Explain how you can tell whether $P$ is above or below the line.

12. Explain why corresponding angles of similar polygons are necessarily the same size.

13. (Continuation) If all the angles of a triangle are equal in size to the angles of another triangle, the triangles are similar. Justify this statement. Is this the converse of the preceding?

14. Pat has $48 and Kim has $24, so the ratio of Pat's money to Kim's money is 2:1. If they both spend $5, is the ratio still 2:1? Explain how they could spend their money so that the ratio of Pat's money to Kim's money remains 2:1.
1. Suppose that \(ABCD\) is a trapezoid, with \(AB\) parallel to \(CD\), and diagonals \(AC\) and \(BD\) intersecting at \(P\). Explain why
   (a) \(\triangle\)s \(ABC\) and \(ABD\) have the same area;
   (b) \(\triangle\)s \(BCP\) and \(DAP\) have the same area;
   (c) \(\triangle\)s \(ABP\) and \(CDP\) are similar;
   (d) \(\triangle\)s \(BCP\) and \(DAP\) need not be similar.

2. Find the size of the acute angle formed by the intersecting lines \(3x + 2y = 12\) and \(x - 2y = -2\), to the nearest tenth of a degree. Do you need to find the intersection point?

3. Let \(A = (0,5,0)\), \(B = (-2,1,0)\), \(C = (6,-1,0)\), and \(P = (2,3,8)\). Let \(A', B', C'\) be the midpoints of segments \(PA, PB,\) and \(PC\), respectively. Make a diagram, and describe the relationship between triangle \(ABC\) and its image \(A'B'C'\).

4. Write an equation that says that \(P = (x, y)\) is 5 units from \((0,0)\). Plot several such points. What is the configuration of all such points called? How many are lattice points?

5. The midpoints of the sides of a quadrilateral are joined to form a new quadrilateral. For the new quadrilateral to be a rectangle, what must be true of the original quadrilateral?

6. Given the line whose equation is \(y = 2x + 3\) and the points \(A = (0,0)\), \(B = (1,9)\), \(C = (2,5)\), \(D = (3,3)\), and \(E = (4,10)\), do the following:
   (a) Plot the line and the points on the same axes.
   (b) Let \(A'\) be the point on the line that has the same \(x\)-coordinate as \(A\). Subtract the \(y\)-coordinate of \(A'\) from the \(y\)-coordinate of \(A\). The result is called the residual of \(A\).
   (c) Calculate the other four residuals.
   (d) What does a residual tell you about the relation between a point and the line?

7. The area of an equilateral triangle with \(m\)-inch sides is 8 square inches. What is the area of a regular hexagon that has \(m\)-inch sides?

8. A parallelogram has 10-inch and 18-inch sides and an area of 144 square inches.
   (a) How far apart are the 18-inch sides?  (b) How far apart are the 10-inch sides?
   (c) What are the angles of the parallelogram?  (d) How long are the diagonals?

9. Let \(P = (1.35,4.26)\), \(Q = (5.81,5.76)\), \(R = (19.63,9.71)\), and \(R' = (19.63,y)\), where \(R'\) is on the line through \(P\) and \(Q\). Calculate the residual value \(9.71 - y\).

10. (Continuation)
    (a) Given that \(Q' = (5.81,y)\) is on the line through \(P\) and \(R\), find \(y\). Calculate \(5.76 - y\).
    (b) Given that \(P' = (1.35,y)\) is on the line through \(Q\) and \(R\), find \(y\). Calculate \(4.26 - y\).
    (c) Which of the three lines best fits the given data? Why do you think so?

11. Write an equation that describes all the points on the circle whose center is at the origin and whose radius is \(r\).
1. Campbell is about to attempt a 30-foot putt on a level surface. The hole is 4 inches in diameter. Remembering the advice of a golf pro, Campbell aims for a mark that is 6 inches from the ball and on the line from the center of the hole to the center of the ball. Campbell misses the mark by a sixteenth of an inch. Does the ball go in the hole?

2. A trapezoid has 11-inch and 25-inch parallel sides, and an area of 216 square inches. (a) How far apart are the parallel sides? (b) If one of the non-parallel sides is 13 inches long, how long is the other one? (N.B. There are two answers to this question. It is best to make a separate diagram for each.)

3. Graph the circle whose equation is \( x^2 + y^2 = 64 \). What is its radius? What do the equations \( x^2 + y^2 = 1, \ x^2 + y^2 = 40, \) and \( x^2 + y^2 = k \) all have in common? How do they differ?

4. Taylor lets out 120 meters of kite string, then wonders how high the kite has risen. Taylor is able to calculate the answer, after using a protractor to measure the 63-degree angle of elevation that the string makes with the ground. How high is the kite, to the nearest meter? What (unrealistic) assumptions did you make in answering this question?

5. Find the sine of a 45-degree angle. Use your calculator only to check your answer.

6. A triangle that has a 5-inch and a 6-inch side can be similar to a triangle that has a 4-inch and an 8-inch side. Find an example. Check that your example really is a triangle.

7. Let \( A = (1, 5), \ B = (3, 1), \ C = (5, 4), \ A' = (5, 9), \ B' = (11, -3), \) and \( C' = (17, 6) \). Show that there is a dilation that transforms triangle \( ABC \) onto triangle \( A'B'C' \). In other words, find the dilation center and the magnification factor.

8. (Continuation) Calculate the areas of triangles \( ABC \) and \( A'B'C' \). Are your answers related in a predictable way?

9. The vectors \([12, 0]\) and \([3, 4]\) form a parallelogram. Find the lengths of its altitudes.

10. The vertices of triangle \( ABC \) are \( A = (-5, -12), \ B = (5, -12), \) and \( C = (5, 12) \). Confirm that the circumcenter of \( ABC \) lies at the origin. What is an equation for the circumcircle?

11. If the sides of a triangle are 13, 14, and 15 cm long, then the altitude drawn to the 14-cm side is 12 cm long. How long are the other two altitudes? Which side has the longest altitude?

12. (Continuation) How long are the altitudes of a triangle whose sides are 26, 28, and 30 cm long? What happens to the area of a triangle if its dimensions are doubled?

13. Find the length of the bisector of the smallest angle of a 3-4-5 triangle.
1. Show the lines \((x, y, z) = (5 + 2t, 3 + 2t, 1 - t)\) and \((x, y, z) = (13 - 3r, 13 + 2r, 4 + 4r)\) are not parallel, and that they do not intersect. Such lines are called skew.

2. Let \(A = (6, 0)\), \(B = (0, 8)\), \(C = (0, 0)\). In triangle \(ABC\), let \(F\) be the foot of the altitude drawn from \(C\) to side \(AB\).
   (a) Explain why the angles of triangles \(ABC\), \(CBF\), and \(ACF\) are the same.
   (b) Find coordinates for \(F\), and use them to calculate the exact lengths \(FA\), \(FB\), and \(FC\).
   (c) Compare the sides of triangle \(ABC\) with the sides of triangle \(ACF\). What do you notice?

3. A similarity transformation is a geometric transformation that uniformly multiplies distances, in the following sense: For some positive number \(m\), and any two points \(A\) and \(B\) and their respective images \(A'\) and \(B'\), the distance from \(A'\) to \(B'\) is \(m\) times the distance from \(A\) to \(B\). You have recently shown that any dilation \(T(x, y) = (mx, my)\) is a similarity transformation. Is it true that the transformation \(T(x, y) = (3x, 2y)\) is a similarity transformation? Explain.

4. The area of the triangle determined by the vectors \(v\) and \(w\) is 5. What is the area of the triangle determined by the vectors \(2v\) and \(3w\)? Justify your answer. Do not assume that \(v\) and \(w\) are perpendicular.

5. Decide whether the transformation \(T(x, y) = (4x - 3y, 3x + 4y)\) is a similarity transformation. If so, what is the multiplier \(m\)?

6. A rectangular sheet of paper is 21 cm wide. When it is folded in half, with the crease running parallel to the 21-cm sides, the resulting rectangle is the same shape as the unfolded sheet. Find the length of the sheet, to the nearest tenth of a cm. Note: in many places outside of the United States, such as Europe, the shape of notebook paper is determined by this similarity property.

7. How much evidence is needed to be sure that two triangles are similar?

8. A line of slope \(\frac{1}{2}\) intersects a line of slope 3. Find the size of the acute angle that these lines form.

9. Square \(ABCD\) has 8-inch sides, \(M\) is the midpoint of \(BC\), and \(N\) is the intersection of \(AM\) and diagonal \(BD\). Find the lengths of \(NB\), \(NM\), \(NA\), and \(ND\).

10. Parallelogram \(PQRS\) has \(PQ = 8\) cm, \(QR = 9\) cm, and diagonal \(QS = 10\) cm. Mark \(F\) on \(RS\), exactly 5 cm from \(S\). Let \(T\) be the intersection of \(PF\) and \(QS\). Find the lengths \(TS\) and \(TQ\).

11. The parallel sides of a trapezoid are 12 inches and 18 inches long. The non-parallel sides meet when one is extended 9 inches and the other is extended 16 inches. How long are the non-parallel sides of this trapezoid?
1. The lengths of the sides of triangle $ABC$ are often abbreviated by writing $a = BC$, $b = CA$, and $c = AB$. Notice that lower-case sides oppose upper-case vertices. Suppose now that angle $BCA$ is right, so that $a^2 + b^2 = c^2$. Let $F$ be the foot of the perpendicular drawn from $C$ to the hypotenuse $AB$. If $a = 5$, $b = 12$ and $c = 13$, what are the lengths of $FA$, $FB$, and $FC$? Does $c = FA + FB$?

2. Verify that $P = (-1.15, 0.97)$, $Q = (3.22, 2.75)$, and $R = (9.21, 10.68)$ are not collinear. (a) Let $Q' = (3.22, y)$ be the point on the line through $P$ and $R$ that has the same $x$-coordinate as $Q$ has. Find $y$, then calculate the residual value $2.75 - y$. (b) Because the segment $PR$ seems to provide the most accurate slope, one might regard $PR$ as the line that best fits the given data. The point $Q$ has as yet played no part in this decision, however. Find an equation for the line that is parallel to $PR$ and that makes the sum of the three residuals zero. Based on these criteria, this is a line of best fit.

3. Apply the Angle-Bisector Theorem to the smallest angle of the right triangle whose sides are 1, 2, and $\sqrt{3}$. The side of length 1 is divided by the bisector into segments of what lengths? Check your answer by asking your calculator for the tangent of a 15-degree angle.

4. Sketch the circle whose equation is $x^2 + y^2 = 100$. Using the same system of coordinate axes, graph the line $x + 3y = 10$, which should intersect the circle twice — at $A = (10, 0)$ and at another point $B$ in the second quadrant. Estimate the coordinates of $B$. Now use algebra to find them exactly. Segment $AB$ is called a chord of the circle.

5. (Continuation) What percentage of the circumference of the circle lies above the chord $AB$? First estimate the percentage, then find a way of calculating it precisely.

6. (Continuation) Find coordinates for a point $C$ on the circle that makes chords $AB$ and $AC$ have equal length. What percentage of the circumference lies below chord $AC$?

7. What is the radius of the smallest circle that surrounds a 5-by-12 rectangle?


9. Without doing any calculation, what can you say about the tangent of a $k$-degree angle, when $k$ is a number between 90 and 180? Explain your response, then check with your calculator.

10. Draw a right triangle that has an 18-cm hypotenuse and a 70-degree angle. To within 0.1 cm, measure the leg adjacent to the 70-degree angle, and express your answer as a percentage of the hypotenuse. Compare your answer with the value obtained from your calculator when you enter $\text{COS} 70$ in degree mode. This is an example of the cosine ratio.