Advanced Study in Calculus BC

Summer 2016 Assignment

Instructions
1. Complete the accompanying problem sets on separate sheets of paper from this packet.
2. Start each problem set on a new sheet of paper.
3. Skip a line between problems.
4. Use a pencil and a good eraser. Erase (or start over) rather than cross out.
5. Show your work clearly and neatly. This includes the Multiple Choice problems.
6. Provide brief verbal explanations when appropriate.

And, most importantly
7. **DO NOT USE A CALCULATOR.** In many problems using a calculator would bypass the intended fact, theorem, or computational technique being assessed.

Mail your solutions (**not this packet**) to me before August 20. If you can produce a high quality readable scanned copy you may e-mail that to me, but I prefer the original by regular mail.

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Problem Set 4. Limits

1. Calculate \( \lim_{x \to 2} (2x^2 - 6x + 1) \).
2. Find the limit: \( \lim_{x \to 1} f(x) \) if \( f(x) = \begin{cases} x^2 + 4, & x \neq 1 \\ 2, & x = 1 \end{cases} \).
3. Find the limit: \( \lim_{x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \).
4. Find the limit: \( \lim_{x \to 0} \frac{x}{\sin 3x} \).
5. Determine the value of \( c \) so that \( f(x) \) is continuous on the entire real line if \( f(x) = \begin{cases} x^2, & x \leq 3 \\ 5, & x > 3 \end{cases} \).
6. Find all vertical asymptotes of \( f(x) \) if \( f(x) = \frac{2x - 2}{(x - 1)(x^2 + x - 1)} \).
1. If \( f(x) = 2x^2 + 4 \), which of the following will calculate the derivative of \( f(x) \)?

(a) \( \frac{[2(x + \Delta x)^2 + 4] - (2x^2 + 4)}{\Delta x} \)

(b) \( \lim_{\Delta x \to 0} \frac{(2x^2 + 4 + \Delta x) - (2x^2 + 4)}{\Delta x} \)

(c) \( \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2] + 4 - (2x^2 + 4)}{\Delta x} \)

(d) \( \frac{(2x^2 + 4 + \Delta x) - (2x^2 + 4)}{\Delta x} \)

(e) None of these

2. Differentiate: \( y = \frac{1 + \cos x}{1 - \cos x} \)

(a) \(-1\)

(b) \(-2 \csc x\)

(c) \(2 \csc x\)

(d) \(-2 \sin x\)

(e) None of these

3. Find \( dy/dx \) for \( y = x^3 \sqrt{x} + 1 \).

(a) \( \frac{3x^2 + x^{3/2}}{2x + 1} \)

(b) \( \frac{3(3x + 6)}{2\sqrt{x} + 1} \)

(c) \(3x^2 \sqrt{x} + 1\)

(d) None of these

(e) None of these

4. Find \( f'(x) \) for \( f(x) = (2x^2 + 5)^3 \).

(a) \(7(4x)^6\)

(b) \(4(2x^2 + 5)^2\)

(c) \(28x(2x^2 + 5)^6\)

(d) None of these

(e) None of these

5. Find \( dy/dx \) for \( y = \frac{5 + 3}{x - 1} \)

(a) 0

(b) \(-3\)

(c) \(-4\)

(d) None of these

(e) None of these

6. The position equation for the movement of a particle is given by \( x = (t^2 - 1)^5 \) when \( x \) is measured in feet and \( t \) is measured in seconds. Find the acceleration at two seconds.

(a) 342 \text{ units/sec}^2

(b) 18 \text{ units/sec}^2

(c) 288 \text{ units/sec}^2

(d) 10 \text{ units/sec}^2

(e) None of these

7. Find \( \frac{dy}{dx} \) if \( y^3 - 3xy + x^3 = 7 \).

(a) \( \frac{2x + y}{3x - 2y} \)

(b) \( \frac{3y - 2x}{2y - 3x} \)

(c) \( \frac{2x}{3 - 2y} \)

(d) \( \frac{2x}{y} \)

(e) None of these

8. Find \( y' \) if \( y = \sin(x + y) \).

(a) \( \frac{\cos(x + y)}{1 - \cos(x + y)} \)

(b) \( \sin(x + y) \)

(c) \( \cos(x + y) \)

(d) 1

(e) None of these

9. Differentiate: \( y = \sec^2 x + \tan^2 x \).

(a) \( 0 \)

(b) \( \tan x + \sec^2 x \)

(c) \( \sec^2 x \)(\sec^2 x + \tan^2 x) \)

(d) None of these

(e) None of these

10. Find the derivative: \( s(t) = \csc \frac{t}{2} \)

(a) \( -\csc \frac{t}{2} \cot \frac{t}{2} \)

(b) \( \frac{1}{2} \cot \frac{t}{2} \)

(c) \( \frac{1}{2} \csc \frac{t}{2} \cot \frac{t}{2} \)

(d) \( \frac{1}{2} \csc \frac{t}{2} \cot \frac{t}{2} \)

(e) None of these

11. Find an equation for the tangent line to the graph of \( f(x) = 2x^3 - 2x + 3 \) at the point where \( x = 1 \).

(a) \( y = 2x - 2 \)

(b) \( y = 4x^2 - 6x + 5 \)

(c) \( y = 2x + 1 \)

(d) \( y = 4x^2 - 6x + 2 \)

(e) None of these

12. Find all points on the graph of \( f(x) = -x^3 + 3x^2 - 2 \) at which there is a horizontal tangent line.

(a) \( (0, -2), (2, 2) \)

(b) \( (0, -2) \)

(c) \( (1, 0), (0, -2) \)

(d) \( (2, 2) \)

(e) None of these
Problem Set 6. Applications of the Derivative

1. Find all open intervals on which the function \( f(x) = \frac{x^2}{x^2 + 4} \) is decreasing.
   (a) (0, \infty)         (b) (-2, 2)         (c) (-\infty, 0)
   (d) (-\infty, \infty)   (e) None of these

2. Find the values of \( x \) that give relative extrema for the function \( f(x) = 3x^3 - 5x^2 \).
   (a) Relative maximum: \( x = 0 \); Relative minimum: \( x = \sqrt[3]{\frac{3}{5}} \)
   (b) Relative maximum: \( x = -1 \); Relative minimum: \( x = 0 \)
   (c) Relative maximum: \( x = \pm 1 \); Relative minimum: \( x = 0 \)
   (d) Relative maximum: \( x = 0 \); Relative minimum: \( x = \pm 1 \)
   (e) None of these

3. Find all intervals on which the graph of the function is concave upward: \( f(x) = \frac{x^2 + 1}{x^2} \).
   (a) (-\infty, \infty)         (b) (-\infty, -1) and (1, \infty)         (c) (-\infty, 0) and (0, \infty)
   (d) (1, \infty)             (e) None of these

4. Which of the following functions has a horizontal asymptote at \( y = 2 \)?
   (a) \( x - \frac{2}{3x - 5} \)         (b) \( \frac{2x}{\sqrt{x} - 2} \)         (c) \( \frac{2x^2 - 6x + 1}{1 + x^2} \)
   (d) \( \frac{2x + 1}{x^2 + 1} \)       (e) None of these

5. Find all points of inflection: \( f(x) = \frac{1}{5} x^4 - 2x^2 + 15 \).
   (a) (2, 0)         (b) (2, 0), (-2, 0)         (c) (0, 15)
   (d) (2, \frac{16}{5}), (-2, \frac{16}{5})       (e) None of these

6. The management of a large store wishes to add a fenced-in rectangular storage yard of 20,000 square feet, using the building as one side of the yard. Find the minimum amount of fencing that must be used to enclose the remaining 3 sides of the yard.
   (a) 400 ft         (b) 200 ft         (c) 20,000 ft
   (d) 500 ft       (e) None of these

7. State why the Mean Value Theorem does not apply to the function \( f(x) = \frac{2}{x + 1} \) on the interval [-3, 0].
   (a) \( f(x) = -3 \neq f(0) \)         (b) \( f \) is not continuous at \( x = -1 \).
   (c) \( f \) is not defined at \( x = -3 \) and \( x = 0 \).
   (d) Both (a) and (b)
   (e) None of these

8. The side of a cube is measured to be 3.0 inches, if the measurement is correct to within 0.01 inch, use differentials to estimate the propagated error in the volume of the cube.
   (a) \( \pm 0.00001 \text{ in.}^3 \)         (b) \( \pm 0.006 \text{ in.}^3 \)         (c) \( \pm 0.027 \text{ in.}^3 \)
   (d) \( \pm 0.27 \text{ in.}^3 \)           (e) None of these

Problem Set 7. Additional Practice Modeled after AP Multiple Choice

C-1
If \( y = (x^2 + 1)^3 \), then \( \frac{dy}{dx} = \)
(A) \( 3x^2(x^2 + 1)^2 \)  (B) \( 6x^3(x^2 + 1) \)  (C) \( 2(3x^2 + 1) \)  (D) \( 3x^4(x^2 + 1) \)  (E) \( 6x^2(x^2 + 1) \)

C-4
If \( y = \frac{2x + 3}{x + 2} \), then \( \frac{dy}{dx} = \)
(A) \( \frac{12x + 13}{(3x + 2)^2} \)  (B) \( \frac{12x - 13}{(3x + 2)^2} \)  (C) \( \frac{5}{(3x + 2)^2} \)  (D) \( \frac{-5}{(3x + 2)^2} \)  (E) \( \frac{2}{3} \)

C-7
The graph of \( f' \), the derivative of the function \( f \), is shown above. Which of the following statements is true about \( f \)?
(A) \( f \) is decreasing for \(-1 \leq x \leq 1 \).
(B) \( f \) is increasing for \(-2 \leq x \leq 0 \).
(C) \( f \) is increasing for \(-1 \leq x \leq 2 \).
(D) \( f \) has a local minimum at \( x = 0 \).
(E) \( f \) is not differentiable at \( x = -1 \) and \( x = 1 \).

C-9
If \( f(x) = \ln(x + 4 + e^x) \), then \( f'(0) \) is
(A) \( -\frac{2}{3} \)  (B) \( \frac{1}{3} \)  (C) \( \frac{1}{4} \)  (D) \( \frac{2}{3} \)  (E) nonexistent
C-15
Let \( f \) be the function with derivative given by \( f'(x) = x^2 - \frac{3}{x} \). On which of the following intervals is \( f \) decreasing?
(A) \((-\infty, -1]\) only
(B) \((-\infty, 0)\)
(C) \([-1, 0)\) only
(D) \((0, \sqrt{3}]\)
(E) \(\left[\sqrt{3}, \infty\right)\)

C-17
Let \( f \) be the function given by \( f(x) = 2x^2 \). The graph of \( f \) is concave down when
(A) \( x < -2 \)  (B) \( x > -2 \)  (C) \( x < -1 \)  (D) \( x > -1 \)  (E) \( x < 0 \)

C-20
\[
    f(x) = \begin{cases} 
        x + 2 & \text{if } x \leq 3 \\
        4x - 7 & \text{if } x > 3 
    \end{cases}
\]

Let \( f \) be the function given above. Which of the following statements are true about \( f \)?
I. \( \lim_{x \to 3^-} f(x) \) exists.
II. \( f \) is continuous at \( x = 3 \).
III. \( f \) is differentiable at \( x = 3 \).
(A) None
(B) I only
(C) II only
(D) I and II only
(E) I, II, and III

C-21
The second derivative of the function \( f \) is given by \( f''(x) = ax - a(x - b)^2 \). The graph of \( f'' \) is shown above. For what values of \( x \) does the graph of \( f \) have a point of inflection?
(A) 0 and \( a \) only  (B) 0 and \( m \) only  (C) \( b \) and \( f/j \) only  (D) 0, \( a \), and \( b \)  (E) \( b/j \), and \( k \)

C-24
Let \( f \) be the function defined by \( f(x) = 4x^2 - 5x + 3 \). Which of the following is an equation of the line tangent to the graph of \( f \) at the point where \( x = -1 \)?
(A) \( y = 7x - 3 \)
(B) \( y = 7x + 7 \)
(C) \( y = 7x + 11 \)
(D) \( y = -5x - 1 \)
(E) \( y = -5x - 5 \)

C-25
A particle moves along the x-axis so that at time \( t \geq 0 \) its position is given by \( x(t) = 2t^3 - 21t^2 + 72t - 53 \). At what time \( t \) is the particle at rest?
(A) \( t = 1 \) only
(B) \( t = 3 \) only
(C) \( t = -\frac{7}{2} \) only
(D) \( t = 3 \) and \( t = \frac{7}{2} \)
(E) \( t = 3 \) and \( t = 4 \)

C-26
What is the slope of the line tangent to the curve \( 3y^2 - 2x^2 = 6 - 2xy \) at the point \((3, 2)\)?
(A) 0  (B) \( \frac{4}{9} \)  (C) \( \frac{7}{9} \)  (D) \( \frac{6}{7} \)  (E) \( \frac{5}{3} \)

C-27
Let \( f \) be the function defined by \( f(x) = x^3 + x \). If \( g(x) = f^{-1}(x) \) and \( g(2) = 1 \), what is the value of \( g''(2) \)?
(A) \( \frac{1}{13} \)  (B) \( \frac{1}{4} \)  (C) \( \frac{7}{4} \)  (D) \( 4 \)  (E) \( 13 \)

C-28
Let \( g \) be a twice-differentiable function with \( g'(x) > 0 \) and \( g''(x) > 0 \) for all real numbers \( x \), such that \( g(4) = 12 \) and \( g(5) = 18 \). Of the following, which is a possible value for \( g(6) \)?
(A) 15  (B) 18  (C) 21  (D) 24  (E) 27
C-79
For which of the following does \( \lim_{x \to a} f(x) \) exist?

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only

C-80
The function \( f \) is continuous for \(-2 \leq x \leq 1\) and differentiable for \(-2 < x < 1\). If \( f(-2) = -5 \) and \( f(1) = 4 \), which of the following statements could be false?

(A) There exists \( c \) where \(-2 < c < 1\), such that \( f'(c) = 0 \).
(B) There exists \( c \), where \(-2 < c < 1\), such that \( f''(c) = 0 \).
(C) There exists \( c \), where \(-2 < c < 1\), such that \( f'(c) = 3 \).
(D) There exists \( c \), where \(-2 < c < 1\), such that \( f''(c) = 3 \).
(E) There exists \( c \), where \(-2 < c < 1\), such that \( f''(c) = f'(c) \) for all \( x \) on the closed interval \(-2 \leq x \leq 1\).

C-81 (You need a calculator for this one)

Let \( f \) be the function with derivative given by \( f'(x) = \sin(x^2 + 1) \). How many relative extrema does \( f \) have on the interval \( 2 < x < 4 \)?

(A) One  (B) Two  (C) Three  (D) Four  (E) Five

C-87 (You need a calculator for this one)

The function \( f \) has first derivative given by \( f'(x) = \frac{x}{1 + x^2} \). What is the \( x \)-coordinate of the inflection point of the graph of \( f \)?

(A) 1.008  (B) 0.473  (C) 0  (D) -0.278  (E) The graph of \( f \) has no inflection point.

C-90
For all \( x \) in the closed interval \([2, 5]\), the function \( f \) has a positive first derivative and a negative second derivative. Which of the following could be a table of values for \( f' \)?